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# Hidden Cayley graph structures

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**Abstract**

A contribution to the study of the structure of complete Cayley graphs is given by means of a method of construction of graphs whose vertices are labeled by toral subgraphs induced by equally colored $K_3$'s. These graphs shed some light on the mentioned structure because the traversal of each one of its edges from one of its end vertices into the other one represents a transformation between corresponding toral embeddings. As a result, a family of labeled graphs indexed on the odd integers appear whose diameters are asymptotically of the order of the square root of the number of vertices. This family can be obtained by modular reduction from a graph arising from the Cayley graph of the group of integers with the natural numbers as set of generators, which have remarkable local symmetry properties.

## 1. Introduction

For every odd $n = 2k + 1 \in \mathscr{Z}$, let $\mathrm{Cay}(\mathscr{Z}_n, \{1, 2, \ldots, k\})$ be the (undirected) complete Cayley graph of the cyclic group $\mathscr{Z}_n$. Its study is undertaken by means of a graph $G_n'$ whose vertices are labeled by the triples $abc$ of distinct elements $a, b, c$ of $\{1, 2, \ldots, k\}$ such that $a + b \in \{-c, c\}$. Equivalently, the vertices of $G_n'$ can be labeled by the toral subgraphs $\mathrm{Cay}_n(abc)$ induced by the triangles whose edges are labeled by colors $a, b, c \in \{1, 2, \ldots, k\}$.

The traversal of each edge $e$ of $G_n'$ from one of its end vertices into the other one represents a transformation between the corresponding triangular toral embeddings consisting in the deletion of the edges corresponding to one color and the addition of the edges corresponding to another color. (Incidentally, after the edge deletions here and before the new edge additions, we have momentarily a 2-colored toral embedding

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of degree and girth equal to 4, that may be taken as the label of $e$.) This is pointed out in the second remark of Section 2.2, subsequent example and the accompanying Fig. 4.

As a result, and by means of a natural extension, a family of algebraically labeled graphs appears. This may be of interest as a source for interconnection network models with special desirable properties. The family is a sequence of graphs $G_{n,3}$, where $n$ ranges over all odd positive integers.

For each odd positive integer $n \geqslant 7$, $G_{n,3}$ is a graph with maximum degree 3. The asymptotic behavior of the diameter of $G_{n,3}$ is of the order of the square root of its number of vertices. This will be seen at the end of Section 2.

The graphs $G_{n,3}$ will be obtained by modular reduction from a corresponding infinite graph $G_{\infty,3}$. This will be of fundamental importance for the establishment of the claimed properties.

A subsequent family of graphs $G_{n,4}$ ($n$ odd) with maximum degree 6 and remarkable local symmetry properties has been treated in [4]. It is shown in [5] that the asymptotic behavior of the diameter of $G_{n,4}$ is of the order of the cubic root of its number of vertices.

## 2. Triangles in complete Cayley graphs

### 2.1. Triangular types

**Definition 2.1.** Let $n=2k+1$ be an odd positive integer. Let $\mathscr{Z}_n$ be the cyclic group of order $n$. Let $\mathrm{Cay}_n = \mathrm{Cay}(\mathscr{Z}_n, I_n)$ be the undirected *Cayley graph* of $\mathscr{Z}_n$ with *generating set* $I_n = \{1,2,\ldots,k\} \subset \mathscr{Z}_n$.

Note that $\mathrm{Cay}_n$ is a *complete graph* $K_n$ edge-colored so that the degree of each color on each vertex is 2.

**Example.** We represent in Fig. 1(a) the Cayley graph $\mathrm{Cay}_7$, for which $k=3$, so that its edges are colored with colors 1, 2 and 3.

**Definition 2.2.** A triangle of $\mathrm{Cay}_n$ is said to have $K_3$-type $(a,b,c)$ if its edges have colors $a,b,c \in I_n$. Here, the order in which the colors $a,b,c$ are written is irrelevant. We will write $abc$ instead of $(a,b,c)$ in case there is no confusion at supressing commas and parentheses.

**Remark.** Notice that the colors $a,b,c$ of a $K_3$-type $abc$ must satisfy necessarily $a+b \in \{c,-c\}$ (or the other relations, like this one, obtained by permuting $a,b,c$). In fact, $a,b,c$ represent the edge colors of actual triangles in $\mathrm{Cay}_n$, due to the algebraic structure of it. This observation allows us to express adequately the generalization of a $K_3$-type given in Definition 2.2 below.

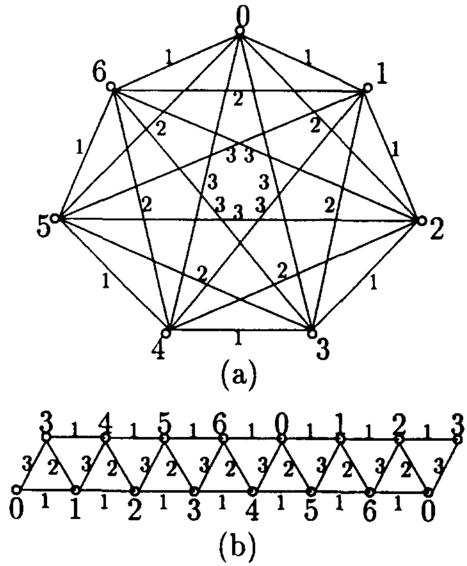


Fig. 1. Standard and toral representations of $\mathrm{Cay}_7$.

**Example.** $\mathrm{Cay}_7$ has 14 triangles of $K_3$-type 123. They define an *embedding* of $\mathrm{Cay}_7$ into a *two-dimensional torus*. This can be seen in the representation of Fig. 1(b). In this figure, the edge with end vertices labeled with 0 and 3 drawn on the left must be identified with the one drawn on the right. Similar identifications must take place in the figure for the pairs of horizontal edges having equal end-vertex labels. This is an example of Proposition 2.6, for which we have Definition 2.3.

**Definition 2.3.** A complete subgraph (in particular a triangle) of $\mathrm{Cay}_n$ is said to be *TMC* or *totally multicolored* if its edges have different colors.

**Example.** The TMC $K_3$-types in:
(a) $\mathrm{Cay}_9$ are 123, 134 and 234.
(b) $\mathrm{Cay}_{11}$ are 123, 134, 145, 235 and 245.
(c) $\mathrm{Cay}_{13}$ are 123, 134, 145, 156, 235, 246, 256 and 346.

**Lemma 2.4.** *Each TMC triangle $t$ of* $\mathrm{Cay}_n$ *and edge $e$ of $t$ determine exactly one TMC triangle $t' \neq t$ of* $\mathrm{Cay}_n$ *with the same edge colors of $t$ and sharing $e$ with $t$.*

**Definition 2.5.** A *toral embedding* of a graph $G$ is a representation of $G$ in the surface of a torus $T$, the unique closed surface of genus 1 (see, for example, [8]). A toral embedding of $G$ is said to be *triangular* (respectively, *quadrangular*) if all the faces determined by $G$ on $T$ are bounded by triangles (respectively, 4-cycles) of $G$.

**Proposition 2.6.** *Given an odd positive integer $n=2k+1$ and colors $a,b,c\in I_n$ forming a TMC triangle and such that* $\gcd(a,b,c,n)=1$, *there are exactly $2n$ triangles of $K_3$-type abc. These triangles define a triangular toral embedding of the subgraph* $\mathrm{Cay}_n(abc)$ *of* $\mathrm{Cay}_n$ *generated by the edges colored with $a,b,c\in I_n$.*

**Proof.** Let $abc$ be a TMC $K_3$-type. Consider all simultaneous translations of the vertex labels of a triangle representing a $K_3$-type $abc$. We get $n$ triangles representing $abc$. The other $n$ triangles that complete the $2n$ ones mentioned in the statement are obtained by simultaneous change of sign of the vertex labels of the first $n$ triangles we just got. The claimed toral embedding depends on the preceding lemma and the following argument, provided that $\gcd(a,b,c,n)=1$. The edges labeled with colors $a$ and $b$ in $\mathrm{Cay}_n$ span a graph $\mathrm{Cay}_n(ab)$ that can be embedded in the torus by means of a quadrangular toral embedding, as follows: There exists a sequence of $n$-cycles in $\mathrm{Cay}_n$ of the following form:

$$C_0=(0,a,2a,\dots,(n-1)a),\qquad C_1=(b,a+b,2a+b,\dots,(n-1)a+b),$$

$$C_2=(2b,a+2b,\dots,(n-1)a+2b),\qquad C_3=(3b,a+3b,\dots,(n-1)a+3b),\dots$$

Since $\mathrm{Cay}_n(ab)$ is a finite graph, then the edge $0a$ must be repeated more than once in this sequence of cycles. Say that this happens in the $m$th cycle of the sequence, where $m>1$, which then must coincide with the cycle $C_0$ (possibly with a rotation of the positions of the vertices of the cycle). Thus, the edges in the cycles $C_0,C_1,C_2,\dots,C_{m-1}$ can be represented horizontally as in the middle graph of Fig. 4, comprising all the edges labeled with color $a$ and repetition on the top of the representations for the edges of $C_0$. The edges labeled with color $b$ can be represented vertically, with parallel repetition of the vertices and edges only on the left and right rims of the resulting quadrangular representation. By identifying the corresponding vertices and edges repeated twice on these rims, we first get a finite cylinder whose boundary is formed by twice $C_0$. Now, by identifying the corresponding vertices and edges of $C_0$ repeated twice, we arrive at our quadrangular toral representation of $\mathrm{Cay}_n(ab)$. We can now transform this embedding into an embedding of $\mathrm{Cay}_n(abc)$. This is done by adding in each 4-cycle of the embedding an edge of color $c$, which is uniquely determined in each case, so that all the edges of color $c$ form a class of parallel edges, too. (The reader may notice a certain resemblance of part of the above argument with the standard proof of Lagrange theorem in elementary group theory). $\square$

**Remark.** Let $n=2k+1$ be an odd positive integer. Let $a,b,c\in I_n$ be colors forming a $K_3$-type $abc$. If, in addition, there exists $2<r\in\mathscr{Z}$ such that $n=rn'$, $a=ra'$, $b=rb'$ and $c=rc'$, with $\gcd(a,b,c,n)=r$, (so that $\gcd(a',b',c',n')=1$), then Proposition 2.6 can be generalized: $\mathrm{Cay}_n(abc)$ has $r$ ($\geqslant 3$) components. Each of these components admits a TMC triangular toral embedding of its own totaling $r$ such embeddings. It can be seen

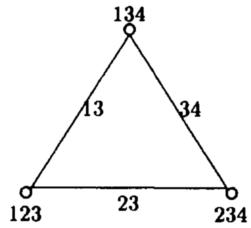


Fig. 2. A representation of $G'_9$.

that these $r$ components can be embedded altogether into a closed surface of genus $r$, but not of lesser genus.

### 2.2. Graphs associated to $K_3$-types

**Definition 2.7.** Given $n = 2k + 1 \geqslant 7$, we define a graph $G'_n$: its vertices are (labeled by) the TMC $K_3$-types $abc$ of $\mathrm{Cay}_n$; any two such vertices are adjacent if their labels share two colors. If such vertices are $abc$ and $abd$, for example, then the resulting edge is (labeled by) $(a,b)$ or $ab$.

**Example.** Fig. 2 shows a representation of $G'_9$, which is a triangle with alternate vertex and edge labels in the following order:

$$123, 13, 134, 34, 234, 23, 123.$$

For example, the edge between the vertices (labeled by the $K_3$-types) $123 = (1,2,3)$ and $134 = (1,3,4)$ is labeled by $13 = (1,3)$. Analogously for the other two edges. Similarly, it can be seen that $G'_{11}$ is a 5-cycle with alternate vertex and edge labels in the following order:

$$123, 13, 134, 14, 145, 45, 245, 25, 235, 23, 123.$$

Fig. 3 shows a representation of $G'_{13}$. This suggests the presence of a quite natural orbital representation of the quotient group $\mathscr{W}_{13}$ obtained from the group $\mathscr{U}_{13} = \mathscr{Z}_{13} - \{0\}$ of units of $\mathscr{Z}_{13}$ by identifying units having opposite signs. This quotient group is isomorphic to $\mathscr{Z}_6$ and is generated, for example, by the pair $\{-2,2\}$, the class of $2 \in \mathscr{U}_{13}$. If each class in $\{-u,u\} \subseteq \mathscr{U}_{13}$ is represented by $u$, where $0 < u < \frac{13}{2}$, then the generator 2 (that is $\{-2,2\}$) yields the only orbit $\{1,2,4,5,3,6\}$ in $\mathscr{W}_{13}$. Fig. 3 shows the resulting vertex label orbits of $G'_{13}$: $\{123, 246, 451, 532, 364, 615\}$ and $\{134, 265\}$. It also shows the resulting edge-label orbits: $\{13, 26, 41, 52, 34, 65\}$ and $\{23, 46, 51\}$.

**Remark.** The example of Fig. 3 illustrates a tool that is suitable for every odd prime $n$ once we get a primitive root of $\mathscr{Z}_n$, like the primitive root 2 of $\mathscr{Z}_{13}$, and consider its

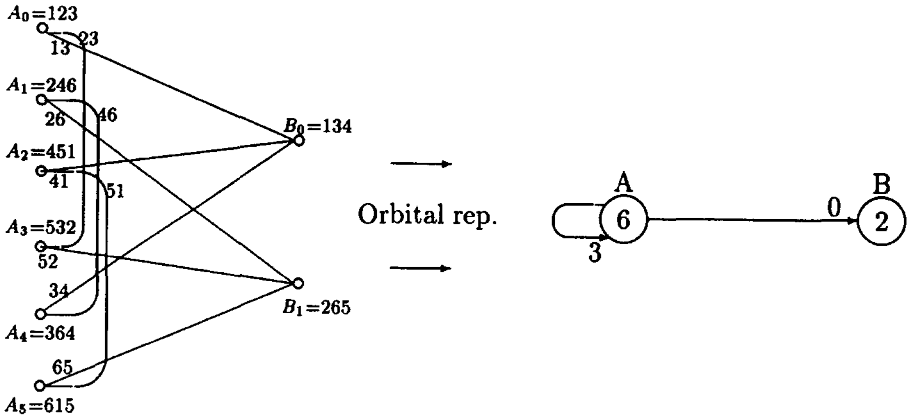


Fig. 3. Orbital representation of $G'_{13}$.

image in $\mathscr{W}_n$ as a generator. This furnishes us with a convenient orbital representation of $G'_n$, see [7].

**Remark.** The graph $G'_n$ of Definition 2.7 sheds light on the hidden structure of $\mathrm{Cay}_n$ as follows. A vertex of $G'_n$ labeled with $abc$ such that $\gcd(a,b,c)=1$ represents a toral embedding of $\mathrm{Cay}_n(abc)$. Let $e$ be an edge of $G'_n$ between vertices labeled $abc$ and $abd$. Then the traversal of $e$ from $abc$ to $abd$ represents the transformation from $\mathrm{Cay}_n(abc)$ to $\mathrm{Cay}_n(abd)$ given by the deletion of the edges colored $c$ and the addition of the edges colored $d$. It may be also be taken to represent a transformation between the corresponding toral embeddings.

**Example.** Let $e$ be the edge $13=\{1,3\}$ in $\mathrm{Cay}_9$. Consider the traversal of $e$ in the direction indicated in Fig. 4(a), that is from end vertex 123 to end vertex 134. This traversal encodes the changes produced on the toral representation of $\mathrm{Cay}_9(123)$ (suggested in Fig. 4(b)), by means of the following steps:
(1) elimination of the edges labeled with color 2 in $\mathrm{Cay}_9(123)$; this yields a toral embedding $\eta$ of the subgraph $\mathrm{Cay}_9(1,3)$ induced by the edges labeled with colors 1 and 3 (see Fig. 4(c));
(2) addition to $\eta$ of the edges labeled with color 4; this yields a toral embedding of $\mathrm{Cay}_9(134)$ (see Fig. 4(d)).

The following two definitions generalize the definitions of $K_3$-type and of $G'_n$ given above, in a convenient way for our purposes.

**Definition 2.8.** A $K_3$-type $\bmod n$ is a triple (multiset) $abc$ with $a,b,c \in I_n \cup \{0\}$ (i.e. formed by three unordered entries in $I_n$, possibly with repetitions) and such that the sum of two of these elements of $I_n$, say $a+b$, equals $c$ or $-c$.

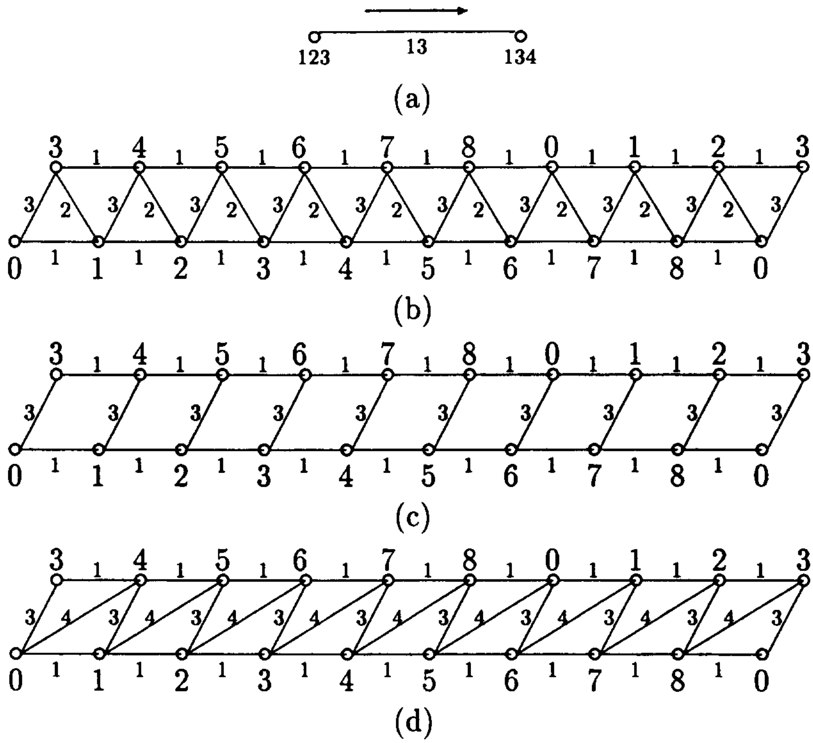


Fig. 4. How the traversal of an edge of $\mathrm{Cay}_9$ encodes subgraph changes in $\mathrm{Cay}_9$.

**Definition 2.9.** We define $G_n$ as the graph whose vertices are (labeled by) the $K_3$-types $\bmod n$ (not only those which are TMC) and adjacency given as in Definition 2.8, where each edge of $G_n$ is considered with multiplicity one.

**Lemma 2.10.** *If $n$ is an odd integer $\geqslant 7$, then $G_n$ is a supergraph of $G'_n$.*

**Proof.** This is clear, considering that $G'_n$ has multiplicity one for each one of its edges. □

**Example.** Fig. 5 depicts the graph $G_9$, which differs from $G'_9$ in that it consists of three connected components:

(a) A connected supergraph of $G'_9$, as can be seen by comparing Figs. 2 and 5. Each of the vertices of this component has a new path of length 2 in $G_9$ arriving to vertices of $G'_9$ of the form $(a,2a,3a)$. Here $a$ is relatively prime to 9 in $I_9$, i.e. $a = 1,2,4$. The other two vertex labels in this path are of the forms $(a,a,2a)$ and $(a,a,0)$. They have degrees 2 and 1 in $G_9$, respectively.

(b) An image of $G_3$ with all the labels of $I_3$ in it transformed into labels of $I_9$ by (integer) multiplication by $\frac{9}{3} = 3$.

(c) An isolated vertex with label 000.

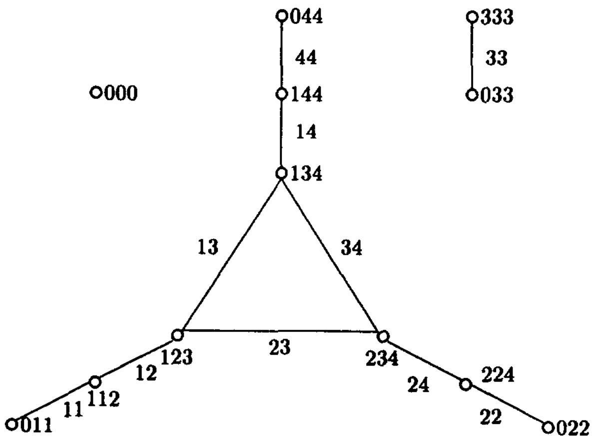


Fig. 5. Representation of $G_9$.

This illustrates the following proposition.

**Proposition 2.11.** *The following properties of $G'_n$ and $G_n$ hold*:

(1) *A vertex $v$ of $G'_n$ labeled with the $K_3$-type abc has either degree* 2 *or* 3, *depending on whether its label abc is of the form $(d,2d,3d)$, for some $d \in I_n$ relatively prime to $n$, or not. If the degree here is* 2, *then $v$ has an extension path of length* 2 *in $G_n$, with associated sequence of labels of alternating vertices and edges*:

$$(d,2d,3d),(d,2d),(d,d,2d),(d,d),(d,d,0).$$

(2) *The graphs $G_n$ and $G'_n$ are not necessarily connected. In fact, components containing the vertex labeled* 123 *do exist, for $n$ nonprime. Moreover, if $1<m<n$, where $m$ divides $n$, then there exists a component of $G_n$ containing the $K_3$-type $(m,2m,3m)$ but not* 123. *Furthermore, $G_n$ is composed by exactly one connected component for each divisor $m$ of $n$, with $1\leqslant m<n$. There is a graph isomorphism from each such component into the correspondent $G_{n/m}$ given by simultaneous division of all labeling colors in the component by $m$.*

**Proof.** (1) Each vertex $abc$ of $G'_n$ gives rise to three different labels of edges of $G_n$, namely $ab$, $bc$ and $ca$. If $abc=(d,2d,3d)$ for some $d\in I_n$ relatively prime to $n$, then one of these edges is $(d,2d)$. But $(d,2d)$ is an edge of $G_n$ and not of $G'_n$, since its other end vertex is not TMC: $(d,d,2d)$, that has only one other neighbor in $G_n$: $0dd$. This yields a path of length 2 attached to $G'_n$ at $(d,2d,3d)$.

(2) All the vertices $abc$ of $G'_n$ with $\gcd(a,b,c)=m$, where $m$ divides $n$, can be taken by repeated application of the relation of adjacency of $G_n$ into the vertex $(m,2m,3m)$. This is done by reducing at each step the sum of the three component numbers of the $K_3$-type considered in turn. This criterion applies separately for each positive divisor

$m<n$ of $n$. This yields in each case a graph isomorphic to $G_{n/m}$. In fact, all the labels of such a component of $G_n$ may be divided simultaneously by $m$. □

**Definition 2.12.** For $n\geqslant 7$, we define $G_{n,3}$ as the component of $G_n$ containing the vertex labeled 123.

**Remark.** We are mostly interested in the component $G_{n,3}$ of $G_n$, since all the other components of $G_n$ can be seen to be isomorphic to graphs of the form $G_{m,3}$, where $1<m<n$ and $m$ divides $n$.

**Example.** The graph $G_{27}$ has as components: $G_{27,3}$ and the three components of $G_9$ depicted in Fig. 5, but with all their labels simultaneously multiplied by $\frac{27}{9}=3$.

The graph $G_{\infty,3}$ presented in the following definition allows to generate all the graphs $G_{n,3}$, as we will see in the following example.

**Definition 2.13.** Let $\mathscr{N}=I_\infty=\{0,1,2,\ldots\}$ be the set of the nonnegative integers considered as a set of colors. A $K_3$-type (over $\mathscr{X}$) is a triple (multiset) $(a,b,c)=abc$ with $a,b,c\in I_\infty$ and such that the (integer) sum of the two smallest of these colors, $a+b$ if $a\leqslant b\leqslant c$, equals the largest, $c$. We define now the graph $G_{\infty,3}$: its vertices are labeled by the $K_3$-types $abc$ over $\mathscr{X}$ with $\gcd(a,b,c)=1$; any two such vertices are adjacent if their labels share exactly two integers; each edge here is considered with multiplicity one.

**Example.** Fig. 6 shows how the graph $G_{\infty,3}$ can be generated algorithmically in the plane: First, $G_{\infty,3}$ is initially provided with a 2-path $P_0$ whose alternate vertex and edge labels (that we depict running downwards) are $011, 11, 112, 12, 123$. From the vertex labeled 123 downwards, each vertex labeled, say, with $abc$ (including $abc=123$, too) is provided with two children: a left child labeled $(a,c,a+c)$ and a right child labeled $(b,c,b+c)$. In this case, the corresponding connecting labels are $ac$ and $bc$. This algorithmic generation of $G_{\infty,3}$ is justified by the following proposition.

(In Fig. 6 and in other figures ahead, hexadecimal notation and its continuation in the alphabet are used.)

**Proposition 2.14.** *The graph $G_{\infty,3}$ is an infinite tree. Moreover, consider the graph*

$$G_{\infty,3}-\{011,11,112,12\}$$

*obtained by removing from $G_{\infty,3}$ the edges labeled with* 11 *and* 12 *and the vertices labeled with* 011 *and* 112. *Then, $G_{\infty,3}$ can be represented as a binary tree in the plane by assigning to each vertex $abc$ in it a left child $(a,c,a+c)$ and a right child $(b,c,b+c)$.*

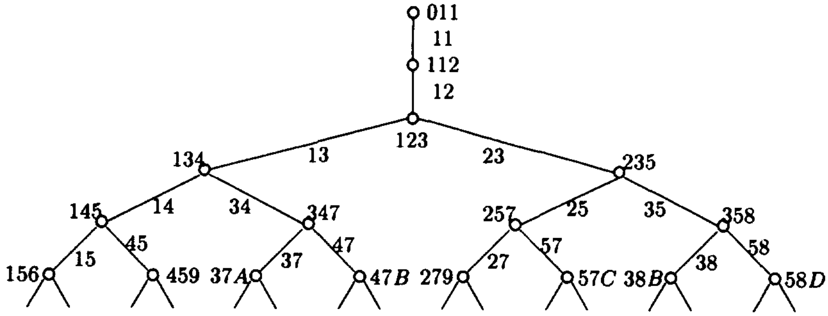


Fig. 6. Algorithmic generation of $G_{\infty,3}$.

**Proof.** Consider a vertex $abc$ in $G_{\infty,3} - \{011, 11, 112, 12\}$ with $a<b<c$. A path from $abc$ to 123 can be found as follows. The standard algorithm to determine the greatest common divisor of $a$ and $b$ (which of course is equal to 1) yields successively the relations

$$\begin{array}{lllll} b & = q_1 a & + r_1, & 0 \leqslant r_1 & < a, \\ a & = q_2 r_1 & + r_2, & 0 \leqslant r_2 & < r_1, \\ r_1 & = q_3 r_2 & + r_3, & 0 \leqslant r_3 & < r_2, \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ r_{k-2} & = q_k r_{k-1} & + r_k, & 0 \leqslant r_k & < r_{k-1}, \\ r_k & = q_{k+1} r_k & + 0 & \cdot & \end{array} \qquad (1)$$

These can be seen to produce a path $\xi$ the initial step of whose development is as follows:

(1) $\xi$ starts at $abc$;
(2) $\xi$ goes upward in the plane successively by means of $q_1$ left-child edges (i.e. to the right!) unless $q_1 = 1$;
(3) if $q_1 = 1$, then $\xi$ goes upward by means of only one right-child edge (i.e. to the left!).

This way $\xi$ arrives at the vertex $(r_1, a, a + r_1)$ in $G_{\infty,3}$. If $(r_1, a, a + r_1) = 123$, this gives a whole description of $\xi$. Otherwise, an inductive step (that we omit here) which is essentially of a similar nature to the detailed initial one above takes place. This inductive step applied $k$ times upwards (one per equation in (1) above, in descending order), yields our claimed arrival to vertex 123. □

**Example.** To generate any particular $G_{n,3}$, where $n$ is an odd positive integer, it is sufficient to proceed as follows:

1. take a large connected subgraph containing $P_0$;
2. then reducing each color to its corresponding color in $I_n$, where each color $i \in I_n$ represents the class $\{i, -i\}$ of oppositely signed integers $i$, $-i \bmod n$.

We need to formalize this concept of reduction $\bmod n$, for this and further purposes.

**Definition 2.15.** A one-to-one correspondence $\Theta_n$ from $\mathcal{N} = I_\infty$ onto $I_n$ such that $m' = \Theta_n(m)$ is defined as follows:

(1) first, we take $m'' \equiv m \bmod n$ such that $0 \leqslant m'' < n$;

(2) if $m'' > n/2$ then we take $m' = n - m''$; otherwise, we take $m' = m''$.

We say that $m'$ is the *reduction* of $m\,\mathrm{MOD}\,n$ (where we write MOD in capital letters).

**Proposition 2.16.** *For an odd positive integer $n \geqslant 7$, the graph $G_{n,3}$ can be obtained from*

(1) *a connected subgraph $F$ of $G_{\infty,3}$ containing $P_0$ and*

(2) *the vertices whose labeling $K_3$-types have composing color numbers less or equal than $n$.*

*By reducing all the colors composing of labels of vertices and edges of $F\,\mathrm{MOD}\,n$, identifications of the resulting vertices and edges occur, yielding a graph identical to $G_{n,3}$.*

**Proof.** Let us assume, without loss of generality, that $F$ is minimal under the conditions given in the statement. Apart from vertex 011 of $F$, the other labels of vertices of $F$ that by reduction $\mathrm{MOD}\,n$ result into $K_3$-types of the form $0aa$ have $n$ as one of its composing colors. Moreover, they are of the form $(n, n-a, a)$, where $0 < a < n$. The only neighbor of each such vertex in $F$ is of the form $(a, n-a, b)$, with $b \neq n$, thus reducing into an image vertex of the form $aab$. Other reductions of vertices of $F$ result into TMC $K_3$-types $\bmod n$, and identifications occur, some of which lead to cycles of $G_{n,3}$. □

**Example.** As an application of Proposition 2.16, the reader may take a connected subgraph $F$ of $G_{\infty,3}$ containing $P_0$ and the vertices whose labeling $K_3$-types have colors $\leqslant 9$. Then proceed to change integer labels to corresponding labels mod 9 to produce, through the appearing identifications, the graph $G_{9,3}$.

### 2.3. *Asymptotics of $K_3$-type graphs*

**Theorem 2.17.** *The number of vertices of $G_{n,3}$ is $\mathrm{O}(n.\phi(n))$, where $\phi(n)$ is the Euler characteristic or totient of $n$.*

**Proof.** Every vertex of the form $aa0$, where $\gcd(a,n) = 1$, belongs to $G_{n,3}$. Thus, there are $\lfloor \phi(n)/2 - 1 \rfloor$ paths whose ends are vertices 011 and $0aa$, where $0 < a \leqslant \lfloor n/2 \rfloor$ and $\gcd(a,n) = 1$. Since the distance from $0aa$ to 011 in $G_{n,3}$ is $\leqslant a$, we have that the number of vertices of $G_{n,3}$ is $\leqslant n.\phi(n)$, yielding our claim. □

In order to prove the claimed asymptotic result for the diameter of $G_{n,3}$, we need the following lemmas.

**Lemma 2.18.** *Let abc be a vertex of $G_{n,3}$. Then $(ka,kb,kc)$ is a vertex of $G_{n,3}$ for every positive integer k such that* $\gcd(k,n)=1$.

**Lemma 2.19.** *If abc is adjacent to abd in $G_{n,3}$, then $(ka,kb,kc)$ is adjacent to $(ka,kb,kd)$ in $G_{n,3}$, for every positive integer k such that* $\gcd(k,n)=1$.

**Theorem 2.20.** *The diameter of $G_{n,3}$ is $\Omega(n)$.*

**Proof.** Let $a$ be a color of $I_n$ with $a\neq 1$ and $\gcd(a,n)=1$. We see that the largest distance among two vertices of $G_{n,3}$ happens between 011 and a vertex of the form $0aa$, because of Proposition 2.16. An easy induction allows to see that the distance from vertex 011 to vertex $(1,a,a+1)$ is $a$. Let $\Phi$ be the transformation that operates by multiplying all the colors composing vertex labels in the path from 011 to $(1,a,a+1)$ by $h$, where $ha=\pm 1 \bmod n$. Then the distance from vertex $(1,a,a+1)$ to vertex $0aa$ can be obtained by means of $\Phi$. By the two previous lemmas, we see that $\Phi$ is an isomorphism between paths. Its image is the path from vertex 011 to vertex $(1,h,h+1)$. This path has length $h$. Thus, the distance from vertex 011 to vertex $0aa$ is $\leqslant a+h$. But $a+h\leqslant \lfloor n/2\rfloor + \lfloor n/2\rfloor \leqslant n$. We see that the diameter of $G_{n,3}$ must be less than or equal to this value $n$, which establishes our claim. □

**Corollary 2.21.** *The order of the diameter of $G_{n,3}$ is asymptotically the square root of its number of vertices.*

**Proof.** Since the number of vertices of $G_{n,3}$ is $O(n.\phi(n))$, it is also $O(n^2)$. This and the previous result yield the conclusion. □

We can determine exactly the number of vertices of $G_{n,3}$. To do so, let each $K_3$-type $abc \bmod n$ be represented with $0\leqslant a,b,c<n/2$ in $\mathscr{X}$. Let $\mathscr{A}_n$ (respectively, $\mathscr{B}_n$) be the subset of TMC $K_3$-types $abc \bmod n$ with $a+b=c$ (respectively, $a+b+c=n$) valid in $\mathscr{X}$. Then the set of vertices of $G_n'$ is $\mathscr{A}\cup\mathscr{B}$. In Fig. 7, $K_3$-type denominations included in ovals correspond to $K_3$-types of the form $aab$, so they are placed solely to delimit the shown ordered formation of the elements of $\mathscr{B}_n$, for small values of odd $n$. Fig. 8 is a simplified continuation of Fig. 7 that eases an inductive counting argument for the cardinality of $\mathscr{B}_n$. In fact, for odd $n\geqslant 9$ let $j=j(n)=\lfloor (n-9)/12\rfloor$ and $\ell=\ell(n)=(n-9-12\times j)/2$. It follows that $0\leqslant \ell\leqslant 5$. From the formations in Figs. 8 and 9 it is not difficult to check that for odd $n\geqslant 9$ the cardinality of $\mathscr{B}_n$ is $B(n)=(3\times j+\ell)\times(j+1)+\delta(n)$, where $\delta(n)=0$ if $1\leqslant\ell\leqslant 5$ and $\delta(n)=1$ if $\ell=0$. Fig. 9 shows an arrangement of $K_3$-types, where arrows relate those $K_3$-types $abc$ with $a\leqslant b\leqslant c$. Using this figure, it can be seen that if $n$ is odd $\geqslant 3$ then the cardinality of $\mathscr{A}_n$ is $A(n)=\lfloor (n-3)/4\rfloor \times \lceil (n-3)/4\rceil$.

**Theorem 2.22.** *Let k be a positive integer $\geqslant 4$, let $n=2k+1$ and let*

$$C(n)=\lfloor n/3\rfloor\cdot(k-2+r_3(n)),$$

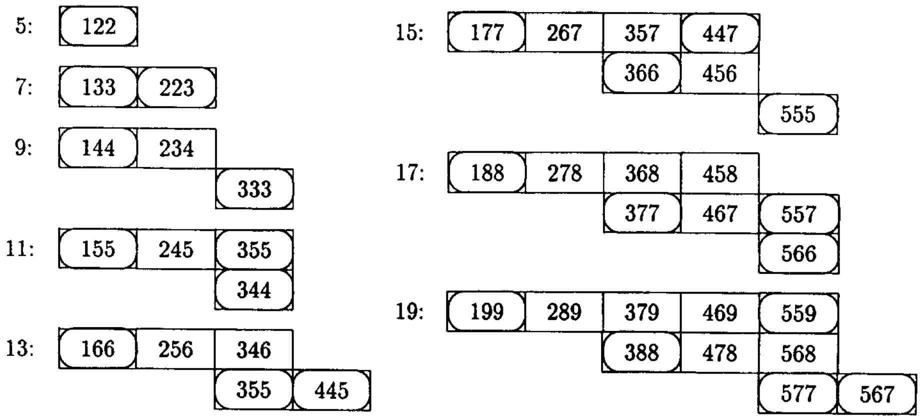


Fig. 7. Generation of the vertex sets $\mathscr{B}_n$.

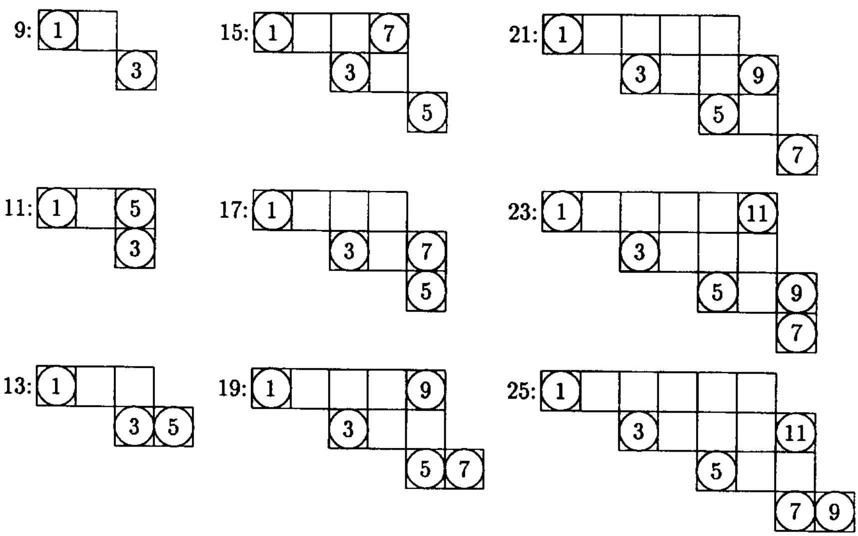


Fig. 8. Simplified form of Fig. 7.

*where $r_3(n)$ is the rest of dividing $n$ by 3. Then the number of vertices of $G_{n,3}$ is $v(n)=\Sigma\{C(m)\colon 1<m\,|\,n\}$, the sum of all $C(m)$ extended over all divisor $m>1$ of $n$.*

**Proof.** It is a matter of calculation to see that $C(n)=A(n)+B(n)$. Then the assertion on odd primes, settled as Corollary 2.23 below, is clear. The rest follows from the principle of inclusion and exclusion. This allows to establish in the first place, for a prime decomposition $n=p_1\times p_2\times\cdots\times p_r$, where $p_1,p_2,\ldots,p_r$ are not necessarily

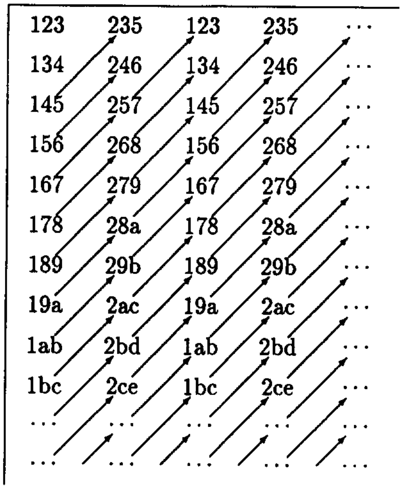


Fig. 9. Generation of the vertex sets $\mathscr{A}_n$.

different, that $v(n)$ equals

$$C(n)-\sum_{i=1}^{r}C(n/p_i)+\sum_{i=1;\,i\neq j}^{r}C(n/p_ip_j)-\sum_{i=1;\,i\neq j;\,i\neq k;\,j\neq k}^{r}C(n/p_ip_jp_k)-\cdots$$

$$+(-1)^{r+1}\sum_{i=1}^{r}C(p_i). \qquad \square$$

**Corollary 2.23.** *If $n$ is a prime odd $\geqslant 9$, then $v(n)=C(n)$.*

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